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ON THE DENSITY OF PROPERLY MAXIMAL CLAIMS IN FINANCIAL MARKETS WITH TRANSACTION COSTS[1]

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We consider trading in a financial market with proportional transaction costs. In the frictionless case, claims are maximal if and only if they are priced by a consistent price process—the equivalent of an equivalent martingale measure. This result fails in the presence of transaction costs. A properly maximal claim is one which does have this property. We show that the properly maximal claims are dense in the set of maximal claims (with the topology of convergence in probability).

1. Introduction. We consider a discrete-time market in d assets with transaction costs. We suppose that $\mathcal{A}$ is the cone of claims attainable from 0 by trading. In [14], following on from Schachermayer [21], Kabanov [15], Kabanov, Stricker and Rasonyi [16] and [17] and many others, Jacka, Berkaoui and Warren showed that if $\mathcal{A}$ is arbitrage-free (i.e., contains no positive elements) then whilst $\mathcal{A}$ may not be closed, its closure [in $L^0(\mathbb{R}^d)$] is also a cone of attainable claims under a new price system and is arbitrage-free if and only if there is a consistent price process for $\mathcal{A}$ (Theorems 3.6 and 4.12). Here, a consistent price process is essentially given by a strictly positive element in the polar cone of $\mathcal{A} \cap L^1$. A consistent price process is a suitable generalization of the concept of the density of an equivalent martingale measure (EMM).

Given a claim $X \in \mathcal{A}$, a standard question is how to hedge it. In other words, how to find a self-financing trading strategy which achieves a final portfolio of X with 0 initial endowment. In the context of frictionless trading,

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this is achieved by seeking maximal claims—claims Y which are maximal in $\mathcal{A}$ with respect to the partial order

$$W \geq X \Leftrightarrow W - X \in \mathbb{R}^d_+ \qquad \text{a.s.},$$

(see [6, 7, 8]). It follows from Kramkov's celebrated result on optional decompositions ([20]) that, at least in a discrete-time context, a claim X in $\mathcal{A}$ is maximal if and only if it is priced at 0 by some EMM. It also follows that this is true if and only if $[\mathcal{A}, X]$, the cone generated by $\mathcal{A}$ and $-X$, is arbitrage-free, in which case its closure is also arbitrage-free.

Consequently (see [13] or [6]), one may obtain a hedging strategy for a maximal claim by martingale representation.

Regrettably, when there are transaction costs, just as $\mathcal{A}$ may be arbitrage-free but $\bar{\mathcal{A}}$ contain an arbitrage, so, in this context, a claim X may be maximal and yet the closure of $[\mathcal{A}, X]$ contain an arbitrage.

In the language of optimization theory, a maximal claim such that the closure of $[\mathcal{A}, X]$ is arbitrage-free is said to be proper efficient with respect to $L^{0,+}$. We shall refer to such claims as properly maximal. We shall show in Theorem 2.9 that a properly maximal claim is priced by some consistent price process and that martingale representation can be used to obtain a hedging strategy. It is then of interest (for hedging purposes) as to whether one can approximate maximal claims by properly maximal claims. This is a problem with a long and distinguished history in optimization theory, going back to [1]. We give a positive answer (up to randomization) in Theorem 4.11: the collection of properly maximal claims is dense in the set of maximal claims.

In a continuous time framework, the problem is more delicate. Indeed, the task of defining a notion of admissible trading strategy, that has a meaningful financial interpretation, is still in progress. A first solution has been given by Kabanov [15], Kabanov and Last [18] and Kabanov and Stricker [19], where the efficient friction assumption was made. More precisely, an admissible self-financing trading strategy was defined as an adapted, vector-valued, cádlág process of finite variation whose increments lie in the corresponding trading/solvency cones and whose terminal value is bounded from below by a constant with respect to the order induced by the terminal solvency cone. Campi and Schachermayer [4] extend these results to bid-ask processes which are not necessarily continuous. In this framework, the discrete-time methodology cannot be adopted, as it is based on the fact that the cone of attainable claims for zero endowment is a *finite* sum of one-period trading cones.

2. Background, notation and preliminary results.

2.1. *Efficient and proper efficient points.* Given a topological vector space Z, a pointed, closed, convex cone C defines a partial order $\overset{C}{\leq}$ on Z by

$$x \overset{C}{\leq} y \Leftrightarrow y - x \in C.$$

For a subset $B \subset Z$, we denote by $\operatorname{cone}(B)$ the cone generated by B, that is,

$$\operatorname{cone}(B) = \{\lambda b : \lambda \in \mathbb{R}^+, b \in B\}.$$

For a convex set $D \subset Z$, we denote by $\operatorname{lin}(D)$ the lineality subspace of D:

$$\operatorname{lin}(D) = \bigcup_{\text{subspaces } V \subseteq D} V,$$

and recall that if D is also a cone then $\operatorname{lin}(D) = D \cap (-D)$.

Definition 2.1. Given a subset $\mathcal{A} \subset Z$, we say that

$$\theta \in \mathcal{A} \text{ is } C\text{-efficient if } \operatorname{cone}(\mathcal{A} - \theta) \cap C = \{0\},$$

and

$$\theta \in \mathcal{A} \text{ is proper } C\text{-efficient if } \overline{\operatorname{cone}(\mathcal{A} - \theta)} \cap C = \{0\}.$$

If the cone C is not pointed then we change the definitions as follows:

$$\theta \in \mathcal{A} \text{ is } C\text{-efficient if } \operatorname{cone}(\mathcal{A} - \theta) \cap C \subset \operatorname{lin}(C),$$

and

$$\theta \in \mathcal{A} \text{ is proper } C\text{-efficient if } \overline{\operatorname{cone}(\mathcal{A} - \theta)} \cap C \subset \operatorname{lin}(C).$$

One of the main problems in multi-criteria optimization theory is to show that each efficient point can be approximated by a sequence of proper efficient points—hereafter we refer to this as the *density problem.*

Remark 2.2. It is easy to see that if $\theta \in \mathcal{A}$ is C'-efficient, with C' a pointed closed convex cone such that $C \setminus \{0\} \subset \operatorname{int}(C')$, then θ is also proper C-efficient.

Take $X \in \overline{\operatorname{cone}(\mathcal{A} - \theta)} \cap C$. If $X \neq 0$ then $X \in \operatorname{int}(C')$ so we can take a neighborhood of X, U, such that $0 \notin U$, $U \cap \operatorname{cone}(\mathcal{A} - \theta) \neq \varnothing$ and $U \subset C'$. But this implies that there is a y with $y \neq 0$, $y \in U \cap \operatorname{cone}(\mathcal{A} - \theta)$ and $y \in C'$ which contradicts the C'-efficiency of θ.

The inverse implication is not always true unless we suppose further conditions on the triplet $(Z, C, \mathcal{A})$.

REMARK 2.3. One way then to solve the density problem is to construct a sequence of pointed closed convex cones $(C_n)_{n\geq 1}$ which decrease to the convex cone C and are such that $C \setminus \{0\} \subset \mathrm{int}(C_n)$. Such a sequence is called a C-*approximating sequence* (or family) of cones. In this case the set $(\theta + C_n) \cap \mathcal{A}$ will converge to the set $(\theta + C) \cap \mathcal{A}$ which is reduced to the singleton $\{\theta\}$ if θ is C-efficient. In consequence, each C_n-efficient point θ_n is proper C-efficient and any sequence of C_n-efficient points will converge to θ.

In [1], Arrow, Barankin and Blackwell solved the density problem in the finite-dimensional case where: $Z = \mathbb{R}^n$, $C = \mathbb{R}^n_+$ with $\mathcal{A}$ a compact, convex set in $\mathbb{R}^n$. This theorem was extended to cover more general topological vector spaces (see [2, 3, 9, 10, 11, 22, 23]). In [22], Sterna-Karwat proved that in a normed vector space Z, there exists a C-approximating sequence of cones if and only if

$$C^{+,i} \stackrel{\text{def}}{=} \{\lambda \in Z^*; \lambda > 0 \text{ on } C \setminus \{0\}\} \neq \varnothing.$$

She then applied this result to the density problem for a compact convex set $\mathcal{A}$.

The case of a locally convex vector space was discussed by, among others, Fu Wantao in [23]. He solved the density problem by supposing that the convex cone C admits a *base* B. This means that B is a convex set, $0 \notin \overline{B}$ and $C = \overline{\mathrm{cone}(B)}$. He used this assumption to construct a C-approximating family of cones.

For a more recent survey of such techniques see [5].

2.2. *Notation and further background.* We are equipped with a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\dots,T}, \mathbb{P})$. We denote the real-valued $\mathcal{F}_t$-measurable functions by $m\mathcal{F}_t$, the nonnegative subset by $m\mathcal{F}_t^+$, the bounded real-valued $\mathcal{F}_t$-measurable functions by $b\mathcal{F}_t$ and the nonnegative subset by $b\mathcal{F}_t^+$. We denote the space of $\mathcal{F}$-measurable random variables in $\mathbb{R}^d$ by $L^0(\mathcal{F}, \mathbb{R}^d)$ (with the metric which corresponds to the topology of convergence in measure) or just L^0_d. And we denote the almost surely nonnegative and nonpositive subsets by $L^{0,+}_d$ and $L^{0,-}_d$ respectively.

DEFINITION 2.4. Through the paper we adopt the following notation. Suppose $F \in \mathcal{F}$, $D \subset L^0_d$ is a convex cone and $\xi \in D$. We define

$$[D, \xi] \stackrel{\text{def}}{=} \mathrm{cone}(D - \xi).$$

Notice that, because D is a cone this satisfies

$$[D, \xi] = D - \mathbb{R}^+ \xi.$$

Moreover, $[D,\xi]$ inherits the convexity property from D.

We define

$$\xi(F) \stackrel{\text{def}}{=} \xi \mathbf{1}_F,$$

where $\mathbf{1}_F$ is the indicator function of F, and

$$D(F) \stackrel{\text{def}}{=} \{\xi(F) : \xi \in D\}.$$

We say that D is *arbitrage-free* if $D \cap L_d^{0,+} = \{0\}$. We denote the complement of a subset B in Ω by B^c.

We recall the setup from Schachermayer's paper [21]: we may trade in d assets at times $0, \dots, T$. We may burn any asset and otherwise trades are on terms given by a bid-ask process π taking values in $\mathbb{R}^{d\times d}$, with π adapted. The bid-ask process gives the (time t) price for one unit of each asset in terms of each other asset, so that

$$\pi_t^{i,i} = 1 \qquad \forall i,$$

and $\pi_t^{i,j}$ is the (random) number of units of asset i which can be traded for one unit of asset j at time t. We assume (with Schachermayer) that we have "netted out" any advantageous trading opportunities, so that, for any t and any $i_0, \dots, i_n$:

$$\pi_t^{i_0,i_n} \le \pi_t^{i_0,i_1} \cdots \pi_t^{i_{n-1},i_n}.$$

The time t trading cone, K_t, consists of all those random trades (including the burning of assets) which are available at time t. Note that Schachermayer refers to this cone as $-\hat{K}_t$. Thus we can think of K_t as consisting of all those random vectors which live (almost surely) in a random closed convex cone $K_t(\omega)$.

Denoting the ith canonical basis vector of $\mathbb{R}^d$ by e_i, $K_t(\omega)$ is defined as the finitely-generated (hence closed) convex cone with generators $\{e_j - \pi_t^{i,j}(\omega)e_i, 1 \le i \ne j \le d;\ \text{and}\ -e_k, 1 \le k \le d\}$. The reader is referred to Theorem 4.5 and the subsequent Remark 4.6 of [14].

We shall say that η is a *self-financing process* if $\eta_t - \eta_{t-1} \in K_t$ for each t, with $\eta_{-1} \stackrel{\text{def}}{=} 0$. We say that $\underline{\xi}$ is a *hedging strategy* if $\underline{\xi} \in K_0 \times \cdots \times K_T$.

It follows that the cone of claims attainable from zero endowment is $K_0 + \cdots + K_T$ and we denote this by $\mathcal{A}$. As we said in the Introduction, $\mathcal{A}$ may be arbitrage-free and yet its closure may contain an arbitrage. However, by Theorem 3.6 of [14], we may and shall assume that if $\bar{\mathcal{A}}$ is arbitrage-free then (by adjusting the bid-ask process) $\mathcal{A}$ is closed and arbitrage-free. We should remark at this point that a very small generalization of this theorem allows us to continue to make this assumption merely if each K_t is

a finitely $\mathcal{F}_t$-generated convex cone with the $\mathcal{F}_t$-measurable generators given by $\Pi_t^1, \dots, \Pi_t^n$: that is,

$$K_t = \left\{ \sum_{i=1}^{n} \alpha_i \Pi_t^i : \alpha_i \in m\mathcal{F}_t^+ \right\}.$$

Henceforth, any such cone will be described as a *finitely $\mathcal{F}_t$-generated cone*.

For any decomposition of $\mathcal{A}$ as a sum of convex cones:

$$\mathcal{A} = M_0 + \cdots + M_t,$$

we call elements of $M_0 \times \cdots \times M_t$, which almost surely sum to 0, *null-strategies* (with respect to the decomposition $M_0 + \cdots + M_t$). We denote the set of null-strategies by $\mathcal{N}(M_0 \times \cdots \times M_t)$. For convenience we denote $K_0 \times \cdots \times K_T$ by $\mathcal{K}$.

In what follows we shall often use (a slight generalization of) Schachermayer's key result (Remark 2.8 after the proof of Theorem 2.1 of [21]):

LEMMA 2.5. *Suppose that*

$$\mathcal{A} = M_0 + \cdots + M_{t-1} + M_t$$

is a decomposition of $\mathcal{A}$ into convex cones with $M_s \subseteq L^0(\mathcal{F}_s, \mathbb{R}^d)$ for $0 \le s \le t-1$, and $b\mathcal{F}_s^+ M_s \subseteq M_s$ for each $s \le t$. If $\mathcal{N}(M_0 \times \cdots \times M_t)$ is a vector space and each M_t is closed, then $\mathcal{A}$ is closed.

REMARK 2.6. Theorem 3.6 of [14] establishes that where $\bar{\mathcal{A}}$ is arbitrage-free, the revised bid-ask process gives rise to finitely generated cones $\tilde{K}_t$ with the further property that $\mathcal{N}(\tilde{\mathcal{K}})$ is a vector space.

COROLLARY 2.7. *There exists a family $(M_t)_{t=0,\dots,T} \subset L_d^0$ with each M_t a closed convex cone $\mathcal{F}_t$-generated by a finite family of $\mathbb{R}^d$-valued $\mathcal{F}_t$-measurable vectors, such that:*

$$\mathcal{A} = M_0 + \cdots + M_T$$

and

$$\mathcal{N}(M_0 \times \cdots \times M_T) = \{\underline{0}\}.$$

PROOF. We have assumed that

$$\mathcal{A} = \tilde{K}_0 + \cdots + \tilde{K}_T,$$

and that $\mathcal{N}(\tilde{\mathcal{K}})$ is a vector space. This implies (by Lemma 2.5) that $\mathcal{A}$ is closed and that $\eta_t \overset{\text{def}}{=} \mathcal{N}(\tilde{K}_t \times \cdots \times \tilde{K}_T)$ is a vector space for each $t = 0, \dots, T-1$. Define ρ_t to be the projection of the closed vector space η_t onto

its first component and define $M_t = \tilde{K}_t \cap \rho_t^\perp$ with $M_T = \tilde{K}_T$. We verify easily that the family $(M_0, \dots, M_T)$ satisfies the conditions of the corollary. $\square$

So, from now on we make the following:

ASSUMPTION 2.8. The cone of claims attainable from 0, $\mathcal{A}$, can be written as $\mathcal{A} = K_0 + \cdots + K_T$ where each K_t is a finitely $\mathcal{F}_t$-generated convex cone and $\mathcal{N}(\mathcal{K}) = \{\underline{0}\}$. Consequently $\mathcal{A}$ is closed.

In what follows, the terms "maximality" and "proper maximality" are defined with respect to the cone $C = L_d^{0,+}$. For more general ordering cones we continue to use the terms "efficiency" and "proper efficiency."

2.3. *Maximal claims and representation.* Recall that in the frictionless setup, $X \in \mathcal{A}$ is maximal if and only if there is an EMM $\mathbb{Q}$ such that $\mathbf{E}_{\mathbb{Q}} X = 0$. Moreover, in that case, denoting the collection of EMMs by Q,

$$\mathbf{E}_{\mathbb{Q}} X = 0$$

for every $\mathbb{Q} \in Q$. In this case, defining V_t as the common value of $\mathbf{E}_{\mathbb{Q}}[X|\mathcal{F}_t]$, the process V is a Q-uniform martingale and hence is representable as a stochastic integral with respect to the discounted price process. See [6] and [13] for details. The stochastic integrand essentially then gives a hedging strategy.

Recall from [21] that the concept of consistent price process is a suitable generalization of the concept of EMM. To be precise, a consistent price process is a martingale $(Z_t)_{0 \le t \le T}$, with Z_t taking values in $K_t^* \setminus \{0\}$, where K_t^* is the polar cone (in $\mathbb{R}^d$) of K_t. The value Z_t plays the same role as the density of the restriction of an EMM to $\mathcal{F}_t$ in the frictionless setting.

THEOREM 2.9. *In the case of transaction costs, suppose that $X \in \mathcal{A}$, then:*

(1) *there exists a consistent price process, Z, such that $\mathbf{E} Z_T \cdot X = 0$ if and only if X is properly maximal.*

(2) *Suppose that X is properly maximal and let Q^Z be the collection of EMMs for the consistent price process Z. Then V^Z, defined by*

$$V_t^Z = \mathbf{E}_{\mathbb{Q}}[Z_T \cdot X|\mathcal{F}_t],$$

is independent of the choice of $\mathbb{Q} \in Q^Z$ and is a Q^Z-uniform martingale.

We recall first Theorem 4.12 of [14], that we will need in the next proof:

$\bar{\mathcal{A}}$, the closure of $\mathcal{A}$ in L^0, is arbitrage-free iff there is a consistent price process Z. In this case, for every strictly positive $\mathcal{F}_T$-measurable $\phi : \Omega \to (0,1]$ we may find a consistent price process Z such that $|Z_T| \leq c\phi$ for some positive constant c.

PROOF OF THEOREM 2.9. First recall Theorem 4.16 of [14]:
If $\theta \in L^0(\mathcal{F}_T, \mathbb{R}^d)$ and $\mathcal{A}$ is closed and arbitrage-free, the following are equivalent:

(i) $\theta \in \mathcal{A}$.
(ii) For every consistent pricing process Z such that the negative part $(\theta \cdot Z_T)^-$ of the random variable $\theta \cdot Z_T$ is integrable, we have

$$\mathbf{E}[\theta \cdot Z_T] \leq 0.$$

Proof of (1): Suppose $X \in \mathcal{A}$ and Z is a consistent price process and X and Z satisfy condition (ii) above. Write X as

$$X = \sum_{s=0}^{T} \xi_s, \tag{2.1}$$

with $\xi_s \in K_s$ for each s, and then, for each t, denote $\sum_{s=0}^{t} \xi_s$ by X_t.

Notice that, since Z is consistent, Z_t is in K_t^* and so $Z_t \cdot \xi_t \leq 0$ for each t. So, in particular,

$$Z_T \cdot X = Z_T \cdot (X_{T-1} + \xi_T) \leq Z_T \cdot X_{T-1}$$

and so

$$(Z_T \cdot X)^- \geq (Z_T \cdot X_{T-1})^-,$$

and thus $(Z_T \cdot X_{T-1})^-$ is integrable. Now, for each t, $X_t \in \mathcal{A}$ (since it is in $K_0 + \cdots + K_t$) so, by part (ii) of Theorem 4.16 of [14],

$$\mathbf{E}[Z_T \cdot X_{T-1}] \leq 0$$

and so, in particular, $Z_T \cdot X_{T-1}$ is integrable and

$$\mathbf{E}[Z_T \cdot X] \leq \mathbf{E}[Z_T \cdot X_{T-1}] = \mathbf{E}[\mathbf{E}[Z_T \cdot X_{T-1} | \mathcal{F}_{T-1}]] = \mathbf{E}[Z_{T-1} \cdot X_{T-1}].$$

Now we iterate the argument [which we may do since $\mathcal{A}_t \stackrel{\text{def}}{=} K_0 + \cdots + K_t$ is closed for each t, which follows from our Assumption 2.8 and $(Z_0, \ldots, Z_t)$ is a consistent price process for $(K_0 \times \cdots \times K_t)$]. We see that if X and Z satisfy the conditions of (ii) above then

$$\mathbf{E}[Z_T \cdot X] = \sum_{s=0}^{T} \mathbf{E}[Z_s \cdot \xi_s]. \tag{2.2}$$

Now, suppose that Z is a consistent price process for $\mathcal{A}$, $X \in \mathcal{A}$ and $\mathbf{E}Z_T \cdot X = 0$. Recalling equation (2.1), it follows from the consistency of Z and (2.2) that

$$Z_t \cdot \xi_t = 0,$$

for each t. Now it is easy to check that $\overline{[\mathcal{A}, X]} = \overline{K_0^{\xi_0} + \cdots + K_T^{\xi_T}}$, where K_t^x denotes the $\mathcal{F}_t$-cone obtained from K_t by adding the generator $-x$. We apply Theorem 4.12 of [14], with $\bar{\mathcal{A}}$ replaced by $\overline{[\mathcal{A}, X]}$ to conclude that $\overline{[\mathcal{A}, X]}$ is arbitrage-free.

Conversely, suppose that $\overline{[\mathcal{A}, X]}$ is arbitrage-free then, by Theorem 4.12 of [14] again, there is a consistent price process, Z, for $K_0^{\xi_0} + \cdots + K_T^{\xi_T}$ satisfying

$$|Z_T| \leq \frac{c}{(1 + \sum_{s=0}^T |\xi_s|_{\mathbb{R}^d})}$$

for some positive c. Notice that, since $Z_t \in (K_t^{\xi_t})^*$ and both ξ_t and $-\xi_t$ are in $K_t^{\xi_t}$, we must have $Z_t \cdot \xi_t = 0$ for each t. It follows, a fortiori, from the fact that Z is consistent for $K_0^{\xi_0} + \cdots + K_T^{\xi_T}$ that Z is consistent for $K_0 + \cdots + K_T$. Notice that $Z_t \cdot \xi_t$ is bounded by c, and hence integrable, for each t and so, by the usual arguments

$$\mathbf{E}Z_T \cdot X = \sum_{t=0}^T \mathbf{E}[Z_t \cdot \xi_t] = 0.$$

To prove (2), simply notice that $V_t^Z = \mathbf{E}_{\mathbb{Q}}[Z_T \cdot X | \mathcal{F}_t] = Z_t \cdot X_t$ by virtue of the usual tower-property arguments and the fact that $\mathbb{Q}$ is an EMM for Z. $\square$

REMARK 2.10. Of course, representation does not guarantee that the "hedging strategy" ξ is admissible: it may be that it is "priced at 0" by Z but is still not in $\mathcal{A}$ because some other consistent price process assigns it a positive price. This can happen if ξ_t is in $\mathrm{span}(K^{\xi_t})$ but not in K^{ξ_t}.

3. An example of a maximal claim which is not proper. We take a simple setup for trading in two assets over two time periods. We set $T = 1$, $d = 2$ and $\Omega = \mathbb{N}$. We take $\mathcal{F}_0$ as the trivial σ-algebra, set $\mathcal{F}_1 = 2^{\mathbb{N}}$ and define $\mathbb{P}$ as any probability measure which puts positive mass on each point of Ω. The bid-ask process π is given by:

$$\pi_0^{1,2} = 1, \qquad \pi_0^{2,1} = k, \qquad \pi_1^{2,1} = 2 \quad \text{and} \quad \pi_1^{1,2} = k,$$

where k will be taken suitably large.

The claim θ is defined by

$$\theta \stackrel{\text{def}}{=} \left(1 - \frac{1}{\omega}\right)e_2 - \left(1 - \frac{1}{2\omega}\right)e_1,$$

which corresponds to the following trading strategy—at time 0 buy 1 unit of asset 2 for one unit of asset 1. At time 1 sell $\frac{1}{\omega}$ units of asset 2 for $\frac{1}{2\omega}$ units of asset 1.

PROPOSITION 3.1. *If k is sufficiently large, the claim θ defined above is maximal but not properly maximal.*

PROOF. Recall from [21] that a *strictly* consistent price process is a martingale $(Z_t)_{0\le t\le T}$ with Z_t taking values in $\mathrm{rint}(K_t^*) \setminus \{0\}$, where rint denotes relative interior. Theorems 1.7 and 2.1 of [21] then show that if there is a *strictly* consistent price process then $\mathcal{A}$ is closed and arbitrage-free.

Now note first that if we take $Z_t = (Z^1, Z^2) = (1, \frac{3}{4})$ for both $t = 0$ and $t = 1$ then Z is a strictly consistent price process because it is clear that Z_t lies in the interior of K_t^* for each t. This follows since $0 < \frac{Z^2}{Z^1} = \frac{3}{4} < \pi^{1,2}$ and $0 < \frac{Z^1}{Z^2} = \frac{4}{3} < \pi^{2,1}$.

Now to show that θ is maximal, suppose that $\phi \in \mathcal{A}$ and $\phi \ge \theta$ a.s. Let

$$\phi = \xi_0 + \xi_1,$$

where $\underline{\xi} \equiv (\xi_0, \xi_1) \in K_0 \times K_1$. It is clear that we may suppose without loss of generality that ξ_0 is either some positive multiple of $e_2 - e_1$ or of $e_1 - ke_2$. Similarly we may suppose that ξ_1 is either some positive $\mathcal{F}_1$-measurable multiple of $e_1 - 2e_2$ or of $e_2 - ke_1$. By taking k sufficiently large we may rule out the second possibility in each case. This leaves us with the case where for suitable $a \in \mathbb{R}^+$ and $B \in m\mathcal{F}_1^+$

$$\xi_0 = a(e_2 - e_1),$$
$$\xi_1 = B(e_1 - 2e_2)$$

and

$$\phi = (a - 2B)e_2 - (a - B)e_1.$$

Now if $\phi \ge \theta$ a.s. then we must have (comparing coefficients of e_2 in ϕ and θ):

$$a - 2B(n) \ge 1 - \frac{1}{n} \qquad \text{for all } n. \tag{3.1}$$

Taking $\limsup_{n\to\infty}$ in (3.1) we see that we must have $a \ge 1$.

However, comparing coefficients of e_1, we must have

$$2(B(n)-a) \geq 2\left(\frac{1}{2n}-1\right) \qquad \text{for all } n, \tag{3.2}$$

and adding (3.1) and (3.2), we see that we must have $a \leq 1$. Hence we see that $a=1$ and so, from (3.1) and (3.2), $B(n)=\frac{1}{2n}$ and $\phi=\theta$. This establishes the maximality of θ.

Now we shall show that $\mathcal{A}^\theta \stackrel{\text{def}}{=} \overline{\text{cone}(\mathcal{A}-\theta)} = \overline{[\mathcal{A},\theta]}$ contains an arbitrage and hence that θ is not proper.

Let ξ denote the strategy above which attains θ, so

$$\xi_0 = (e_2 - e_1)$$

and

$$\xi_1 = \frac{1}{2\omega}(e_1 - 2e_2).$$

Notice that, since $\xi_0 \in \mathcal{A}$, $\psi \stackrel{\text{def}}{=} -\xi_1 = \xi_0 - \theta \in \mathcal{A}^\theta$. It follows, since $(\frac{1}{2}e_1 - e_2) \in K_1$, that $(\frac{n}{\omega}-1)\mathbf{1}_{(\omega\leq n)}(\frac{1}{2}e_1 - e_2) \in \mathcal{A}$ and so (adding $n\psi$)

$$x_n \stackrel{\text{def}}{=} (e_2 - \tfrac{1}{2}e_1)\mathbf{1}_{(\omega\leq n)} \in \mathcal{A}^\theta.$$

Letting $n\to\infty$ we deduce (from the closedness of $\mathcal{A}^\theta$) that $(e_2-\frac{1}{2}e_1)\in\mathcal{A}^\theta$ and so, adding $e_1 - e_2 = -\xi_0 = \xi_1 - \theta \in \mathcal{A}^\theta$, we see that $\frac{1}{2}e_1 \in \mathcal{A}^\theta$, which is an arbitrage. $\square$

4. Some general results and the case $\text{lin}(A) = \{0\}$.

Definition 4.1. We denote by $\mathcal{A}_{t,T}$ the closed cone $K_t + \cdots + K_T$. Given $\theta\in\mathcal{A}$, we denote by $\theta_{t,T}$ the sum $\theta_t + \cdots + \theta_T$.

We say that the decomposition of θ:

$$\theta = \theta_0 + \cdots + \theta_T$$

is a *special decomposition* if, for each $t = 0,\ldots,T-1$,

$$\theta_t \text{ is efficient in } K_t \cap (\theta_{t,T} - \mathcal{A}_{t+1,T}) \text{ with respect to } -\mathcal{A}_{t+1,T}.$$

To be more explicit, the decomposition is special if, for each $t=0,\ldots,T-1$,

(1) $z\in\mathcal{A}_{t+1,T}$
and
(2) $\theta_t - z \in K_t$
together imply that
(3) $z=0$ if $\text{lin}(\mathcal{A}_{t+1,T}) = \{0\}$ or, more generally, $z\in\text{lin}(\mathcal{A}_{t+1,T})$.

Remark 4.2. Notice that a special decomposition is in $\mathcal{K}$.

REMARK 4.3. If we think of hedging a claim, then a special decomposition is "lazy" in that it defers taking action until as late as possible, in some sense.

REMARK 4.4. In the example of Section 3, the decomposition

$$(\theta_0, \theta_1) = (e_2 - e_1, (1/2\omega)(e_1 - 2e_2)),$$

is a special decomposition of $\theta = \theta_0 + \theta_1$.

To show this, we want to prove that θ_0 is efficient in $K_0 \cap (\theta - K_1)$ with respect to the order generated by $-K_1$. So let $\xi_0 \in K_0 \cap (\theta - K_1)$ be such that $\eta_1 = \theta_0 - \xi_0 \in K_1$. Then there exists $a_0, b_0 \in \mathbb{R}^+$ and $a_1, b_1 \in m\mathcal{F}_1^+$ such that

$$\xi_0 = a_0(e_2 - e_1) + b_0(e_1 - ke_2)$$

and

$$\eta_1 = a_1(e_1 - 2e_2) + b_1(e_2 - ke_1).$$

We deduce that

$$b_1 = \frac{1 - a_0 + (2-k)b_0}{2k-1}$$

and

$$a_1 = \frac{(k-1)(-1 + a_0 - (k+1)b_0)}{2k-1}.$$

We take $k \geq 10$ and since $a_1, b_1 \geq 0$ we obtain that $a_0 = 1$ and $b_0 = 0$, which means that $\eta_1 = 0$ and $\theta_0 = \xi_0$. This establishes the desired efficiency of θ_0.

We shall now show that every claim in $\mathcal{A}$ has a special decomposition.

THEOREM 4.5. *Given $\theta \in \mathcal{A}$ there exists a special decomposition of θ.*

REMARK 4.6. To characterize an efficient point, *scalarization* methods are commonly used. One of them consists of considering the following optimization problem

$$\sup\{\lambda(x) : x \in \mathcal{A}\},$$

where, denoting the topological dual of Z by Z^*, $\lambda \in Z^*$ is such that $\lambda \geq 0$ on C and $\lambda > 0$ on $C \setminus \mathrm{lin}(C)$. If the optimum is attained, say at θ, then θ is C-efficient.

To see this, observe that if $x \in \mathrm{cone}(\mathcal{A} - \theta) \cap C$, then we can write it as $x = k(w - \theta)$ for some $k \geq 0$ and $w \in \mathcal{A}$, and since $x \in C$ we see that either $k = 0$ or $w - \theta \in C$ in which case $\lambda(w) \geq \lambda(\theta)$ which implies equality and that $w - \theta \in \mathrm{lin}(C)$.

More generally, for $\xi \in Z$ with $(\xi + C) \cap \mathcal{A} \neq \varnothing$, the arg-max of the optimization problem

$$\sup\{\lambda(x); x \in (\xi + C) \cap \mathcal{A}\},$$

if it exists, is C-efficient.

Proof of Theorem 4.5. The proof uses scalarization.

Notice first that we only need to prove that there is a θ_0 such that

$$\theta_0 \text{ is efficient in } K_0 \cap (\theta - \mathcal{A}_{1,T}) \text{ with respect to } -\mathcal{A}_{1,T},$$

with a general $\mathcal{F}_0$ (not necessarily trivial). This is sufficient, since we may then apply the result to $\theta_{t,T}$ in an inductive argument.

To make the scalarization argument we seek a linear function

$$\lambda : S \to m\mathcal{F}_0,$$

where $S = \operatorname{span}(K_0) = K_0 - K_0$, with the properties that

$$\lambda \leq 0 \qquad \text{a.s. on } C \stackrel{\text{def}}{=} \mathcal{A}_{1,T} \cap S \tag{4.1}$$

and

$$[X \in C \text{ and } \lambda(X) = 0 \text{ a.s.}] \Rightarrow X \in \operatorname{lin}(\mathcal{A}_{1,T}). \tag{4.2}$$

First, notice that $\mathcal{A}_{1,T}$ is closed, and so is S since it is finitely $\mathcal{F}_0$-generated. Thus C is a convex cone, closed in $L^0(\mathcal{F}_0; \mathbb{R}^d)$ and stable under multiplication by elements of $b\mathcal{F}_0^+$. Moreover, since $\mathcal{A}$ is arbitrage-free, so is C.

It follows from the abstract closed convex cone theorem of [14] that there exists a set-valued map $\Lambda : \Omega \to \mathcal{P}(\mathbb{R}^d)$ such that:

(1) Λ is almost surely a closed convex cone;

(2) Λ is Effros–Borel measurable: that is, the event $(\Lambda \cap U \neq \varnothing)$ is in $\mathcal{F}_0$ for any open set $U \subset \mathbb{R}^d$;

(3) $C = \{X \in L^0(\mathcal{F}_0, \mathbb{R}^d) : X \in \Lambda \text{ a.s}\}$.

It is easy to check that the map Λ^*, obtained by defining $\Lambda^*(\omega)$ to be the polar cone of $\Lambda(\omega)$, also satisfies (2) and is also almost surely a closed convex cone. It follows from the fundamental measurability theorem of [12] that there is a countable set $\{Y_n : n \geq 1\}$ in $L^0(\mathcal{F}_0, \mathbb{R}^d)$ such that

$$\Lambda^*(\omega) = \overline{\{Y_n(\omega) : n \geq 1\}}. \tag{4.3}$$

Now we claim that, setting

$$\lambda = \sum 2^{-n} Y_n / |Y_n|_{\mathbb{R}^d},$$

λ satisfies (4.1) and (4.2). Notice first that, since $\Lambda^*(\omega)$ is a.s. a closed convex cone,

$$P(\lambda \in \Lambda^*) = 1. \tag{4.4}$$

To see that λ satisfies (4.1): first take an $X \in C$, then, by property (3), $X \in \Lambda$ a.s. Now, by (4.4), X and λ almost surely lie in polar cones in $\mathbb{R}^d$, so (4.1) holds.

To prove that λ satisfies (4.2): suppose that $X \in C$ and $\lambda \cdot X = 0$ a.s. It follows from the definition of λ and (4.3) that $Y_n \cdot X = 0$ a.s., for each n. We can conclude, again from (4.3), that $\mu \cdot X = 0$ a.s., for any μ such that $\mu \in \Lambda^*$ a.s. Now this in turn implies that $-X$ has the same property, which shows that $-X \in \Lambda$ a.s. We conclude from (3) that $-X \in C$ and hence that $X \in \operatorname{lin}(C)$.

Having obtained our linear function λ which is negative on $C \setminus \operatorname{lin}(C)$, we denote $K_0 \cap (\theta - \mathcal{A}_{1,T})$ by $\hat{K}_0$. Notice that $\hat{K}_0$ is closed since both K_0 and $\mathcal{A}_{1,T}$ are. Now we claim that $\hat{K}_0$ is a.s. bounded, that is, defining $M = \{|X|_{\mathbb{R}^d} : X \in \hat{K}_0\}$:

$$W^* \stackrel{\text{def}}{=} \operatorname{ess\,sup}\{W : W \in M\} < \infty \qquad \text{a.s.} \tag{4.5}$$

To see this, notice first that M is directed upward since, given X and Y in $\hat{K}_0$, $X\mathbf{1}_{(|X|_{\mathbb{R}^d} \geq |Y|_{\mathbb{R}^d})} + Y\mathbf{1}_{(|X|_{\mathbb{R}^d} < |Y|_{\mathbb{R}^d})} \in \hat{K}_0$. It follows that there is a sequence $(X_n)_{n\geq 1} \subset \hat{K}_0$ such that

$$|X_n|_{\mathbb{R}^d} \uparrow W^* \qquad \text{a.s.}$$

Now define F to be the event $(W^* = \infty)$.

Since $X_n \in \hat{K}_0$, there is a $Y_n \in \mathcal{A}_{1,T}$ with $X_n + Y_n = \theta$. Now, setting

$$F_n = F \cap (|X_n|_{\mathbb{R}^d} > 0)$$

and multiplying by $\frac{\mathbf{1}_{F_n}}{|X_n|_{\mathbb{R}^d}}$ we obtain:

$$x_n + y_n = \tilde{\theta}\mathbf{1}_{F_n},$$

where

$$x_n = \mathbf{1}_{F_n} X_n / |X_n|_{\mathbb{R}^d}, \qquad y_n = \mathbf{1}_{F_n} Y_n / |X_n|_{\mathbb{R}^d} \quad \text{and} \quad \tilde{\theta} = \theta / |X_n|_{\mathbb{R}^d}.$$

Now, by Lemma A.2 in [21], we may take a strictly increasing $\mathcal{F}_0$-measurable random subsequence $(\tau_k)_{k\geq 1}$ such that x_{τ_k} converges almost surely, to x say. From the definition of F_n we see that $\tilde{\theta}\mathbf{1}_{F_n} \xrightarrow{\text{a.s.}} 0$ and hence $y_n \xrightarrow{\text{a.s.}} y$ for some $y \in \mathcal{A}_{1,T}$. But this implies that (x, y) is in $\mathcal{N}(K_0 \times \mathcal{A}_{1,T})$ and hence $x = 0$ a.s. However, $|x|_{\mathbb{R}^d} = 1$ a.s. on F and so $\mathbb{P}(F) = 0$, establishing (4.5).

To complete the proof, observe that $\mathbb{L} \stackrel{\text{def}}{=} \{\lambda \cdot X : X \in \hat{K}_0\}$ is also directed upward and so we may take a sequence $(X_n)_{n\geq 1} \subset \hat{K}_0$ such that

$$\lambda \cdot X_n \uparrow l^* \stackrel{\text{def}}{=} \operatorname{ess\,sup} \mathbb{L}.$$

Now we take a strictly increasing $\mathcal{F}_0$-measurable random sequence $(\sigma_k)_{k\geq 1}$ such that X_{σ_k} converges almost surely, to X say, and it follows from (4.5) that $X \in L^0$.

Since $X_n \in \hat{K}_0$, there exists a $Y_n \in \mathcal{A}_{1,T}$ with $\theta = X_n + Y_n$. Now

$$Y_{\sigma_k} = \sum_{i=1}^{\infty} Y_i \mathbf{1}_{(\sigma_k = i)},$$

and $\mathcal{A}_{1,T}$ is a closed convex cone, stable under multiplication by elements of $m\mathcal{F}_0^+$, so $Y_{\sigma_k} \in \mathcal{A}_{1,T}$. Similarly, K_0 is a closed convex cone stable under multiplication by elements of $m\mathcal{F}_0^+$, so it follows that $X_{\sigma_k} \in K_0$ and, since $X_{\sigma_k} + Y_{\sigma_k} = \theta$, that

$$X_{\sigma_k} \in \hat{K}_0.$$

By closure we deduce that $X \in \hat{K}_0$ and $\lambda \cdot X = l^*$ a.s.

Now the scalarization argument shows that we may take $\theta_0 = X$ since if $Y \in \hat{K}_0$ and $X = Y + U$ with $U \in \mathcal{A}_{1,T}$ then

$$\lambda \cdot X = \lambda \cdot Y + \lambda \cdot U,$$

and since $U = X - Y \in \hat{K}_0 - \hat{K}_0 \subset \operatorname{span}(K_0)$ it follows that $U \in \mathcal{A}_{1,T} \cap S$ and so $\lambda \cdot U \leq 0$ a.s. But the maximality of $\lambda \cdot X$ now implies that $\lambda \cdot U = 0$ a.s., and we conclude from (4.1) that $U \in \operatorname{lin}(\mathcal{A}_{1,T})$ which shows that X is efficient. $\square$

We shall now sketch a plan for the main result:

Step (1) take a special decomposition $(\theta_0, \ldots, \theta_T)$ for a maximal claim θ;

Step (2) suppose that there exists a sequence $\mathbb{G} = (G_1, \ldots, G_T)$ such that

$$G_t \in \mathcal{F}_t \qquad \text{for each } t \tag{4.6}$$

and

$$\begin{gathered} \text{whenever } y_t \in K_{t-1} - m\mathcal{F}_{t-1}^+ \theta_{t-1} \text{ with } -y_t(G_t^c) \in A_{t,T} \\ \text{we can conclude that } y_t = 0; \end{gathered} \tag{4.7}$$

Step (3) show that

$$\theta^{\mathbb{G}} \stackrel{\text{def}}{=} \theta_0 + \theta_1(H_1) + \cdots + \theta_T(H_T), \tag{4.8}$$

where $H_t \stackrel{\text{def}}{=} G_1 \cap \cdots \cap G_t$, is properly maximal; to do this, show by backward induction that

$$\theta_{t,T}^{\mathbb{G}} \text{ is properly maximal in } A_{t,T}; \tag{4.9}$$

Step (4) Show that, using randomization, there exists a sequence $(\mathbb{G}^n)_{n\geq 1}$ such that each $\mathbb{G}^n$ satisfies properties (4.6) and (4.7) and $\mathbb{P}(G^n_t)\uparrow 1$ for each t.

For the rest of this section we assume that $\mathrm{lin}(\mathcal{A})=\{0\}$.

We now implement Step (3). For the initial step in the induction we need the following result:

LEMMA 4.7. *Suppose that K is a finitely $\mathcal{F}$-generated convex cone and is arbitrage-free.*

Let $\xi\in K$. Then,

$$\overline{[K,\xi]}=K-m\mathcal{F}^+\xi \tag{4.10}$$

and hence is finitely generated. Moreover, [*using the ordering cone $L^{0,+}_d(\mathcal{F})$*]

$$\xi \text{ is maximal in } K \text{ if and only if it is properly maximal.} \tag{4.11}$$

PROOF. Suppose that $\lambda\in m\mathcal{F}^+$ and define $\lambda_n=\min(\lambda,n)$, then $(n-\lambda_n)\xi\in K$ and hence $-\lambda_n\xi=(n-\lambda_n)\xi-n\xi\in[K,\xi]$. Hence $K-m\mathcal{F}^+\xi\subset\overline{[K,\xi]}$. Conversely, since $K-m\mathcal{F}^+\xi$ is finitely generated it is closed and contains $[K,\xi]$, so (4.10) is satisfied.

To prove (4.11), suppose ξ is maximal in K and that y is an arbitrage in $\overline{[K,\xi]}$, so that $y=x-\alpha\xi$ with $x\in K$ and $\alpha\in m\mathcal{F}^+$. It follows that $\overline{x}\stackrel{\text{def}}{=}\frac{1}{\alpha}x\mathbf{1}_{(\alpha>0)}\in K$ and $\overline{x}=\frac{1}{\alpha}y\mathbf{1}_{(\alpha>0)}+\xi\mathbf{1}_{(\alpha>0)}$. Hence, since $y\geq 0$,

$$z\stackrel{\text{def}}{=}\overline{x}+\xi\mathbf{1}_{(\alpha=0)}\geq\xi$$

and $z\in K$. Since ξ is maximal we get $\overline{x}+\xi\mathbf{1}_{(\alpha=0)}=z=\xi$ and then $y\mathbf{1}_{(\alpha>0)}=0$. Finally, since $\mathbf{1}_{(\alpha=0)}y=\mathbf{1}_{(\alpha=0)}x\in K$ and K is arbitrage-free, we conclude that

$$\mathbf{1}_{(\alpha=0)}y=0$$

and hence that $y=0$. $\square$

THEOREM 4.8. *Suppose that $\mathrm{lin}(\mathcal{A})=\{0\}$, that $\theta\in\mathcal{A}$ is maximal, that $\theta=\theta_0+\cdots+\theta_T$ is a special decomposition of θ, that $\mathbb{G}$ satisfies* (4.6) *and* (4.7) *and that $\theta^{\mathbb{G}}$ is as defined in* (4.8). *Then*

$$\theta^{\mathbb{G}} \text{ is properly maximal in } \mathcal{A}.$$

PROOF. As announced, we shall show that (4.9) holds for each t.

Assume that $\theta^{\mathbb{G}}_{t+1,T}$ is properly maximal in $\mathcal{A}_{t+1,T}$. Now it is easy to check that

$$\overline{[\mathcal{A}_{t,T},\theta^{\mathbb{G}}_{t,T}]}=\overline{\overline{[K_t,\theta^{\mathbb{G}}_t]}+\overline{[\mathcal{A}_{t+1,T},\theta^{\mathbb{G}}_{t+1,T}]}},$$

so if we can show that

$$S_t \stackrel{\text{def}}{=} \overline{[K_t, \theta_t^{\mathbb{G}}]} + \overline{[\mathcal{A}_{t+1,T}, \theta_{t+1,T}^{\mathbb{G}}]}$$

is closed and arbitrage-free then the inductive step is complete. Then Lemma 4.7 gives us the initial step (for S_T).

(S_t *is closed*)

We do this by showing that

$$N \stackrel{\text{def}}{=} \mathcal{N}(\overline{[K_t, \theta_t^{\mathbb{G}}]} \times \overline{[\mathcal{A}_{t+1,T}, \theta_{t+1,T}^{\mathbb{G}}]}) = \{\underline{0}\},$$

and appealing to Lemma 2.5. To do this, notice first that (4.10) tells us that $\overline{[K_t, \theta_t^{\mathbb{G}}]} = K_t - m\mathcal{F}_t^+ \theta_t^{\mathbb{G}} \subset K_t - m\mathcal{F}_t^+ \theta_t$. Now notice that if $z \in \overline{[\mathcal{A}_{t+1,T}, \theta_{t+1,T}^{\mathbb{G}}]}$ then, taking a sequence $z_n \in [\mathcal{A}_{t+1,T}, \theta_{t+1,T}^{\mathbb{G}}]$ converging to z we see that, since $G_{t+1}^c \subset H_{t+1}^c$ and $\theta_{t+1,T}^{\mathbb{G}}$ is supported on H_{t+1}, $z_n \mathbf{1}_{G_{t+1}^c} \in \mathcal{A}_{t+1,T}$ for each n, and hence $z\mathbf{1}_{G_{t+1}^c} \in \mathcal{A}_{t+1,T}$.

So if $(y, z) \in N$ then

$$y(G_{t+1}^c) + z(G_{t+1}^c) = 0$$

and so it follows from (4.7) that $y = 0$ and hence that $z = 0$.

(S_t *is arbitrage-free*)

Suppose that f is an arbitrage in S_t, that is, $f \geq 0$ and $f = y + z$ with $y \in \overline{[K_t, \theta_t^{\mathbb{G}}]}$ and $z \in \overline{[\mathcal{A}_{t+1,T}, \theta_{t+1,T}^{\mathbb{G}}]}$. Then

$$0 = y + (z - f)$$

and so (since $L_d^{0,-} \subset \mathcal{A}_{t+1,T}$), $(y, z - f)$ is in N and so $y = z - f = 0$. It follows that $z = f \geq 0$ and since, by the inductive hypothesis, $\overline{[\mathcal{A}_{t+1,T}, \theta_{t+1,T}^{\mathbb{G}}]} = S_{t+1}$ is arbitrage-free, the inductive step follows. $\square$

We implement the Step (4) of the proof plan as follows:

First define

$$\hat{\Omega}_i = \mathbb{N}, \qquad \hat{\sigma}_t = 2^{\Omega_t}, \qquad \hat{\Omega} = \hat{\Omega}_1 \times \cdots \times \hat{\Omega}_T, \qquad \hat{\mathcal{F}}_t = \sigma_1 \otimes \cdots \otimes \sigma_t,$$

and then define

$$\tilde{\Omega} = \Omega \times \hat{\Omega} \quad \text{and} \quad \tilde{\mathcal{F}}_t = \mathcal{F}_t \otimes \hat{\mathcal{F}}_t.$$

To complete the randomization, define a probability measure $\tilde{\mathbb{P}}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}_T)$ by setting

$$\tilde{\mathbb{P}} = \mathbb{P} \otimes \hat{\mathbb{P}} \otimes \cdots \otimes \hat{\mathbb{P}},$$

where $\hat{\mathbb{P}}$ is the probability measure on $\mathbb{N}$ defined by $\hat{\mathbb{P}}(\{k\}) = 2^{-k}$.

Now set

$$G_t^n = \Omega \times \hat{\Omega}_1 \times \cdots \times \hat{\Omega}_{t-1} \times \{1, \ldots, n\} \times \hat{\Omega}_{t+1} \times \cdots \times \hat{\Omega}_T. \tag{4.12}$$

It is clear that $G_t^n \uparrow \tilde{\Omega}$ as $n \uparrow \infty$ for each t.

We extend then the definition of the cone $\mathcal{A}$ to the new setting by defining $\tilde{K}_t$ to be the convex cone $\tilde{\mathcal{F}}_t$-generated by the same generators as K_t, that is, if

$$K_t = \left\{ \sum_{i=1}^n \alpha_i \Pi_t^i : \alpha_i \in m\mathcal{F}_t^+ \right\}$$

then

$$\tilde{K}_t = \left\{ \sum_{i=1}^n \alpha_i \Pi_t^i : \alpha_i \in m\tilde{\mathcal{F}}_t^+ \right\};$$

and then set $\tilde{\mathcal{A}} = \tilde{K}_0 + \cdots + \tilde{K}_T$.

Lemma 4.9. *Under Assumption 2.8, the convex cone $\tilde{\mathcal{A}}$ is closed, arbitrage-free and the null strategies subset $\mathcal{N}(\tilde{K}_0 \times \cdots \times \tilde{K}_T)$ is trivial. Moreover each maximal claim in $\mathcal{A}$ is also maximal in $\tilde{\mathcal{A}}$.*

Proof. Each property of $\tilde{\mathcal{A}}$ follows from the corresponding property for $\mathcal{A}$ in the same way. So, for example, the null strategies for $\mathcal{A}$ form a vector space, N say. Now take $(\xi_0, \dots, \xi_T) \in \tilde{N}$, where $\tilde{N}$ is the collection of null strategies for $\tilde{\mathcal{K}}$, then fix $(i_1, \dots, i_T) \in \mathbb{N}^T$ then $(\xi_0(\cdot), \xi_1(\cdot; i_1), \dots, \xi_T(\cdot, i_1, \dots, i_T)) \in N$ and so

$$-(\xi_0(\cdot), \xi_1(\cdot; i_1), \dots, \xi_T(\cdot, i_1, \dots, i_T)) \in N,$$

and since $(i_1, \dots, i_T)$ is arbitrary, $-(\xi_0, \dots, \xi_T) \in \tilde{N}$ and hence $\tilde{N}$ is a vector space. The same method—of freezing those arguments of an $\tilde{\mathcal{F}}_t$-measurable random variable which are in $\hat{\Omega}$ will establish each of the results. $\square$

We need one more lemma before we can give the main result:

Lemma 4.10. *If $\theta = \theta_0 + \cdots + \theta_T$ is a special decomposition of $\theta \in \mathcal{A}$, then, for each t, the null strategies subset $\mathcal{N}(\overline{[K_t, \theta_t]} \times \mathcal{A}_{t+1,T})$ is trivial.*

Proof. Since $\theta = \theta_0 + \cdots + \theta_T$ is a special decomposition of θ, it follows that, defining

$$\hat{K}_t = K_t \cap (\theta_{t,T} - \mathcal{A}_{t+1,T}),$$

θ_t is efficient in $\hat{K}_t$ with respect to $-\mathcal{A}_{t+1,T}$, that is,

$$(\hat{K}_t - \theta_t) \cap (-\mathcal{A}_{t+1,T}) = \{0\}. \tag{4.13}$$

Now we know from Lemma 4.7 that $\overline{[K_t, \theta_t]} = K_t - m\mathcal{F}_t^+ \theta_t$, so any null strategy for $\overline{[K_t, \theta_t]} \times A_{t+1,T}$ is of the form $(x_t - \lambda_t \theta_t, x_{t+1,T})$, where $x_t \in K_t$, $\lambda_t \in m\mathcal{F}_t^+$ and $x_{t+1,T} \in A_{t+1,T}$. Now, take such a triple, so that

$$x_t - \lambda_t \theta_t + x_{t+1,T} = 0, \tag{4.14}$$

and multiply (4.14) by $\mathbf{1}_{(\lambda_t=0)}$ to get:

$$x_t \mathbf{1}_{(\lambda_t=0)} + x_{t+1,T} \mathbf{1}_{(\lambda_t=0)} = 0.$$

So, we conclude that

$$x_t \mathbf{1}_{(\lambda_t=0)} = 0, \tag{4.15}$$

because $\mathcal{N}(\mathcal{K}) = \{\underline{0}\}$ and so $\mathcal{N}(K_t \times A_{t+1,T}) = \{\underline{0}\}$.

Now multiply (4.14) by $\alpha_t \overset{\text{def}}{=} \frac{1}{\lambda_t} \mathbf{1}_{(\lambda_t>0)}$ to obtain

$$\alpha_t x_t + \theta_t \mathbf{1}_{(\lambda_t=0)} - \theta_t + \alpha_t x_{t+1,T} = \alpha_t x_t - \theta_t \mathbf{1}_{(\lambda_t>0)} + \alpha_t x_{t+1,T} = 0. \tag{4.16}$$

Now $\alpha_t \in m\mathcal{F}_t^+$ so $\alpha_t x_t \in K_t$ and, since $\theta_t \in K_t$, we see that

$$y_t \overset{\text{def}}{=} \alpha_t x_t + \theta_t \mathbf{1}_{(\lambda_t=0)} \in K_t.$$

Moreover, from (4.16)

$$y_t = \theta_t - \alpha_t x_{t+1,T} = \theta_{t,T} - (\alpha_t x_{t+1,T} + \theta_{t+1,T}),$$

and so

$$y_t \in (\theta_{t,T} - \mathcal{A}_{t+1,T}).$$

We deduce that

$$y_t \in \hat{K}_t.$$

Now $y_t - \theta_t \in \hat{K}_t - \theta_t$ and $y_t - \theta_t \in -\mathcal{A}_{t+1,T}$ so we deduce from (4.13) that

$$y_t - \theta_t = 0 \text{ which implies that } \alpha_t x_t - \theta_t \mathbf{1}_{(\lambda_t>0)} = 0,$$

and, multiplying by λ_t and adding (4.15) we obtain the desired result that

$$x_t - \lambda_t \theta_t = 0. \qquad \square$$

THEOREM 4.11. *Let $\theta \in \mathcal{A}$ be a maximal claim in $\mathcal{A}$ (or indeed in $\tilde{\mathcal{A}}$). Then there exists a sequence of properly maximal claims $(\theta^n)_{n\geq 1}$ in $\tilde{\mathcal{A}}$ which converge a.s. to θ.*

PROOF. Thanks to Lemma 4.9 we may work with $\tilde{\mathcal{A}}$ thoughout. We fix the special decomposition $\theta = \theta_0 + \cdots + \theta_T$ and, taking $\mathbb{G}^n$ as in (4.12), define $\theta^n = \theta^{\mathbb{G}^n}$ using (4.8).

Now suppose that

$$y \in \overline{[\tilde{K}_{t-1}, \theta_{t-1}]} \quad \text{and} \quad -y\mathbf{1}_{(G_t^n)^c} \in \tilde{\mathcal{A}}_{t,T},$$

then, setting $z = -y$,

$$z\mathbf{1}_{(G_t^n)^c} \in \tilde{\mathcal{A}}_{t,T}$$

and

$$y\mathbf{1}_{(G_t^n)^c} + z\mathbf{1}_{(G_t^n)^c} = 0.$$

Now take any $j > n$ then, since y is $\tilde{\mathcal{F}}_{t-1}$-measurable and z is $\tilde{\mathcal{F}}_t$-measurable,

$$y(\omega; \hat{\omega}_1, \ldots, \hat{\omega}_{t-1}) + z(\omega; \hat{\omega}_1, \ldots, \hat{\omega}_{t-1}, j) = 0 \qquad \text{a.s.}$$

Finally, taking $(\hat{\omega}_1, \ldots, \hat{\omega}_{t-1}) = (i_1, \ldots, i_{t-1})$ we see that

$$y(\cdot, i_1, \ldots, i_{t-1}) \in K_{t-1} - m\mathcal{F}_{t-1}^+ \theta_{t-1} = \overline{[K_{t-1}, \theta_{t-1}]}$$

and

$$z(\cdot, i_1, \ldots, i_{t-1}, j) \in \mathcal{A}_{t,T}$$

for each choice of $i_1, \ldots, i_{t-1}, j$ and so it follows from Lemma 4.10 that $y(\cdot, i_1, \ldots, i_{t-1}) = 0$ for each choice of $i_1, \ldots, i_{t-1}, j$ and so $y = 0$. The fact that θ^n is properly maximal now follows from Theorem 4.8. It is obvious that $\theta^n \xrightarrow{\text{a.s.}} \theta$ as $n \uparrow \infty$. $\square$

Remark 4.12. Since the convergence in Theorem 4.11 follows from a truncation, it is clear that if the special decomposition used has the property that $\theta_t \in L^p(\mathcal{F}_t, \mathbb{R}^d)$ for each t then convergence of the properly maximal sequence will also be in L^p by the dominated convergence theorem.

5. The case $\mathbf{lin}(\boldsymbol{\mathcal{A}}) \neq \{\mathbf{0}\}$. In the case where $\text{lin}(\mathcal{A}) \neq \{0\}$ we may still assume that $\mathcal{N}(\mathcal{K}) = \{\underline{0}\}$, however the conclusion of Lemma 4.10 fails, that is, we may no longer conclude that, with θ_t being the tth component of a special decomposition of θ, $\mathcal{N}(\overline{[K_t, \theta_t]} \times \mathcal{A}_{t+1,T}) = \{\underline{0}\}$.

The way around this problem is to focus on $t = 0$ and define $\sim$, an equivalence relation on elements of $K_0 \cap (\theta - \mathcal{A}_{1,T})$, as follows:

$$x \sim y \Leftrightarrow x - y \in \text{lin}(\mathcal{A}_{1,T}).$$

Remark 5.1. Notice that if θ_0 is efficient in $K_0 \cap (\theta - \mathcal{A}_{1,T})$ (with respect to $-\mathcal{A}_{1,T}$), then every element of the equivalence class $[\theta_0]$ is efficient. To see this, take $z \in [\theta_0]$, so $z \in K_0 \cap (\theta - \mathcal{A}_{1,T})$ and $z - \theta_0 \in -\mathcal{A}_{1,T}$.

Now we can easily show that, defining

$$\Sigma_t = m\mathcal{F}_t^+[\theta_t],$$

the correct generalization of Lemma 4.10 holds.

LEMMA 5.2. *If θ_t is efficient, the null space $\mathcal{N}((K_t - \Sigma_t) \times \mathcal{A}_{t+1,T})$ is a vector space.*

PROOF. As indicated, we need only to prove the result in the case where $t = 0$, provided we do not assume that $\mathcal{F}_0$ is trivial.

Suppose $x \in K_0$, $\xi \sim \theta_0$, $\lambda \in m\mathcal{F}_0^+$, $z \in \mathcal{A}_{1,T}$ and

$$x - \lambda\xi + z = 0. \tag{5.1}$$

It is immediate that

$$(x + \xi) - (1 + \lambda)\xi + z = 0, \tag{5.2}$$

and, dividing (5.2) by $1 + \lambda$ we get

$$\tilde{x} - \xi + \tilde{z} = 0.$$

It follows, since $\tilde{x} \in K_0 \cap (\theta - \mathcal{A}_{1,T})$ and ξ is efficient, that $\tilde{z} \in \mathrm{lin}(\mathcal{A}_{1,T})$ and therefore that $\tilde{x} \sim \xi$. And so $\tilde{x} \sim \theta_0$ and $z \in \mathrm{lin}(\mathcal{A}_{1,T})$. So $\xi - \tilde{x} \in K_0 - \Sigma_0$ and, multiplying by $1 + \lambda$,

$$\lambda\xi - x \in K_0 - \Sigma_0$$

and so $\mathcal{N}((K_0 - \Sigma_0) \times \mathcal{A}_{1,T})$ is a vector space. $\square$

We now have another problem since Lemma 4.7 is no longer apparently relevant—at first sight it does not look as though $K_0 - \Sigma_0$ is finitely generated, so it is not clear that it is closed.

LEMMA 5.3. *For each t, there is a $\xi_t \in [\theta_t]$ such that*

$$K_t - \Sigma_t = \overline{[K_t, \xi_t]} = K_t - m\mathcal{F}_t^+\xi_t \tag{5.3}$$

and so $K_t - \Sigma_t$ is closed.

PROOF. As before, we only need to prove the lemma for $t = 0$ and a nontrivial $\mathcal{F}_0$. Now, since $\Sigma_0 \subset K_0$, it is clear that

$$K_0 - \Sigma_0 = K_0 + \Sigma_0 - \Sigma_0. \tag{5.4}$$

We shall prove that, for the right choice of $\xi_0 \in \Sigma_0$,

$$\Sigma_0 - \Sigma_0 = (K_0 - m\mathcal{F}_0^+\xi_0) \cap (m\mathcal{F}_0\xi_0 + \mathrm{lin}(A_{1,T})) = \Sigma_0 - m\mathcal{F}_0^+\xi_0, \tag{5.5}$$

by showing that

$$\Sigma_0 - \Sigma_0 \subset (K_0 - m\mathcal{F}_0^+\xi_0) \cap (m\mathcal{F}_0\xi_0 + \operatorname{lin}(A_{1,T})) \subset \Sigma_0 - m\mathcal{F}_0^+\xi_0. \tag{5.6}$$

Notice that if (5.6) holds then there must be equality throughout, since $m\mathcal{F}_0^+\xi_0 \subset \Sigma_0$, and the result will then follow immediately from (5.4) and (5.5).

We define ξ_0 as follows.

First recall that the generators of K_0 are $(\Pi_0^i)_{1\le i\le m}$. Now define

$$\Phi \stackrel{\text{def}}{=} \left\{(\alpha_1,\dots,\alpha_m) : \sum_i \alpha_i \Pi_0^i \in \Sigma_0; \alpha_i \in m\mathcal{F}_0^+, |\alpha_i| \le 1 \text{ for } i = 1,\dots,m\right\}.$$

It is clear that Φ is a convex set, closed in $L^0(\mathcal{F}_0;\mathbb{R}^m)$.

Now define $p : \Phi \to \mathbb{R}^+$ by

$$p(\underline{\alpha}) = \sum_{i=1}^m \mathbb{P}(\alpha_i > 0).$$

Denote $\sup_{\alpha\in\Phi} p(\alpha)$ by p^* (notice that $p^* \le m$) and take a sequence $(\underline{\alpha_n})_{n\ge1} \subset \Phi$ such that $p(\underline{\alpha_n}) \uparrow p^*$. It follows from the convexity and closure of Φ that

$$\sum_{k=1}^n 2^{-k}\underline{\alpha_k} + 2^{-n}\underline{\alpha_{n+1}} \xrightarrow{\text{a.s.}} \sum_{k=1}^\infty 2^{-k}\underline{\alpha_k} \stackrel{\text{def}}{=} \underline{\hat{\alpha}} \in \Phi$$

and

$$p(\hat{\alpha}) = p^*.$$

Now define

$$\xi_0 \stackrel{\text{def}}{=} \sum_{i=1}^m \hat{\alpha}_i \Pi_0^i.$$

The convexity and closure of Φ ensures that $\xi_0 \in \Sigma_0$. Notice that it follows from the definition of $\hat{\alpha}$ that if $x = \sum_{i=1}^m \alpha_i \Pi_0^i \in \Sigma_0$ then

$$\mathbb{P}((\alpha_i > 0) \cap (\hat{\alpha}_i = 0)) = 0 \qquad \text{for each } i.$$

Denote the middle term in (5.5) by R.

$(\Sigma_0 - \Sigma_0 \subset R)$

Since $\Sigma_0 \subset K_0$ and $\psi \sim \phi \Rightarrow \psi - \phi \in \operatorname{lin}(\mathcal{A}_{1,T})$, which implies that

$$\Sigma_0 \subset (m\mathcal{F}_0\xi_0 + \operatorname{lin}(A_{1,T})),$$

we see that

$$\Sigma_0 \subset R.$$

Now take $x \in \Sigma_0$, so $x \in K_0$ and $z \stackrel{\text{def}}{=} x - \alpha^+\xi_0 \in \text{lin}(A_{1,T})$ for some $\alpha^+ \in m\mathcal{F}_0^+$. It follows that $-x = -z - \alpha^+\xi_0$ so $-x \in (m\mathcal{F}_0\xi_0 + \text{lin}(A_{1,T}))$. All that remains for this step is to prove that

$$-x \in (K_0 - m\mathcal{F}_0^+\xi_0). \tag{5.7}$$

Recall that ξ_0 has maximal support in Σ_0, so if $x = \sum_i \alpha_i \Pi_0^i$ and $\xi_0 = \sum_i \hat{\alpha}_i \Pi_0^i$ then $\beta \stackrel{\text{def}}{=} \max_i\{\frac{\alpha_i}{\hat{\alpha}_i}\} < \infty$ a.s. Since $\beta \in m\mathcal{F}_0^+$ it follows that

$$\beta\xi_0 - x \in K_0$$

and hence, expressing $-x$ as $(\beta\xi_0 - x) - \beta\xi_0$, we conclude that (5.7) holds.

$(R \subset \Sigma_0 - m\mathcal{F}_0^+\xi_0)$

Take $y \in R$. Since $y \in (K_0 - m\mathcal{F}_0^+\xi_0)$ we may write it as

$$y = x - \alpha^+\xi_0,$$

with $x \in K_0$ and $\alpha^+ \in m\mathcal{F}_0^+$. Moreover, since $y \in (m\mathcal{F}_0\xi_0 + \text{lin}(\mathcal{A}_{1,T}))$ we may write it as

$$y = \gamma\xi_0 + z,$$

with $\gamma \in m\mathcal{F}_0$ and $z \in \text{lin}(\mathcal{A}_{1,T})$. Denoting the positive and negative parts of γ by γ^+ and γ^- respectively, it follows that

$$x + \gamma^-\xi_0 = (\gamma^+ + \alpha^+)\xi_0 + z. \tag{5.8}$$

The right-hand side of (5.8) is clearly in $(m\mathcal{F}_0^+\xi_0 + \text{lin}(\mathcal{A}_{1,T}))$ and the left-hand side is clearly in K_0, so we conclude that the common value, w say, is in Σ_0.

Finally, observe that

$$y = w - (\alpha^+ + \gamma^-)\xi_0,$$

and so $y \in \Sigma_0 - m\mathcal{F}_0^+\xi_0$. □

Now we suitably generalize condition (4.7) and Theorem 4.8.

Suppose that $\theta \in \mathcal{A}$ and it is decomposed as $\theta = \theta_0 + \cdots + \theta_T$.

Definition 5.4. For each t, define N_t as the projection onto the first component of the nullspace $\mathcal{N}((K_t - \Sigma_t) \times \mathcal{A}_{t+1,T})$ and define $N_t^\perp$ as the orthogonal complement of N_t (this is well defined thanks to Lemma 5.2 and Lemma A.4 in [21]).

Further suppose that there exists a sequence $\mathbb{G} = (G_1, \ldots, G_T)$ such that

$$G_t \in \mathcal{F}_t \qquad \text{for each } t \tag{5.9}$$

and

$$\begin{gathered} \text{whenever } y_t \in (K_{t-1} - m\mathcal{F}_{t-1}^+[\theta_{t-1}]) \cap N_{t-1}^\perp \\ \text{with } -y_t(G_t^c) \in \mathcal{A}_{t,T} \text{ we may conclude that } y_t = 0. \end{gathered} \tag{5.10}$$

THEOREM 5.5. *Suppose that $\theta \in \mathcal{A}$ is maximal, that $\theta = \xi_0 + \cdots + \xi_T$ is a special decomposition of θ with $\underline{\xi}$ as in Lemma 5.3, so that*

$$K_t - m\mathcal{F}_t^+[\xi_t] = K_t - m\mathcal{F}_t^+\xi_t.$$

Suppose, in addition, that $\mathbb{G}$ satisfies (5.9) and (5.10) and that $\theta^{\mathbb{G}}$ is as defined in (4.8), then

$$\theta^{\mathbb{G}} \text{ is properly maximal in } \mathcal{A}.$$

PROOF. The argument mirrors the proof of Theorem 4.8.

As before we need to show that

$$S_t \stackrel{\text{def}}{=} \overline{[K_t, \xi_t(H_t)]} \cap N_t^{\perp} + \overline{[\mathcal{A}_{t+1,T}, \theta^{\mathbb{G}}_{t+1,T}]}$$

is closed and arbitrage-free.

(S_t *is closed*)

We do this by showing that

$$N \stackrel{\text{def}}{=} \mathcal{N}(([K_t, \xi_t(H_t)] \cap N_t^{\perp}) \times \overline{[\mathcal{A}_{t+1,T}, \theta^{\mathbb{G}}_{t+1,T}]}) = \{\underline{0}\}.$$

To do this, notice first that (4.10) and (5.3) tell us that

$$\overline{[K_t, \xi_t(H_t)]} = K_t - m\mathcal{F}_t^+[\xi_t(H_t)] \subset K_t - m\mathcal{F}_t^+\xi_t.$$

Now notice that, as before, if $z \in \overline{[\mathcal{A}_{t+1,T}, \theta^{\mathbb{G}}_{t+1,T}]}$ then, $z\mathbf{1}_{G^c_{t+1}} \in \mathcal{A}_{t+1,T}$.

So if $(y, z) \in N$ then

$$y(G^c_{t+1}) + z(G^c_{t+1}) = 0$$

and so it follows from (5.10) that $y = 0$ and hence that $z = 0$.

(S_t *is arbitrage-free*)

The argument is unchanged. □

The proof of the revised version of Theorem 4.11 is essentially unchanged. Since the statement does not involve $\mathrm{lin}(\mathcal{A})$ we do not repeat it.

6. Further comments. A slight modification of Theorem 5.5 states, under some mild assumptions, that for any maximal claim $\theta \in \mathcal{A}$, there exists a sequence of properly maximal claims θ_n which converges to θ in probability.

THEOREM 6.1. *Given $\theta \in \mathcal{A}$ is maximal, take a special decomposition of $\theta : \theta = \xi_0 + \cdots + \xi_T$, with $\underline{\xi}$ as in Lemma 5.3, so that*

$$K_t - m\mathcal{F}_t^+[\xi_t] = K_t - m\mathcal{F}_t^+\xi_t.$$

Suppose there exists a sequence $\mathbb{G}^n$ satisfying (5.9) and (5.10), with each G^n_t converging to Ω.

Now define the sequence $\theta^n \stackrel{\text{def}}{=} \theta^{\mathbb{G}^n}$ *as in* (4.8). *Then*

$$\text{the sequence } \theta^n \text{ is properly maximal in } \mathcal{A}$$

and

$$\theta_n \to \theta \qquad \text{in probability.}$$

Unfortunately we are unable to construct such a sequence $\mathbb{G}^n$ in a general setting. We have adopted a randomization approach that allows us to construct such sequence.

We remark that hedging such a randomized sequence is still possible "in the market without randomization." By this we mean that, since trades in the randomized market take place at the same bid-ask prices as in the original market, an individual trader may perform the randomizations and hedge accordingly in the original market.

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